CS5733 Program Synthesis

#14. Type Driven Synthesis

Ashish Mishra, September 24, 2024

Module II

Behavioral constraints

assertions
types
pre/post-conditions

+ bounded guarantees

+ unbounded guarantees

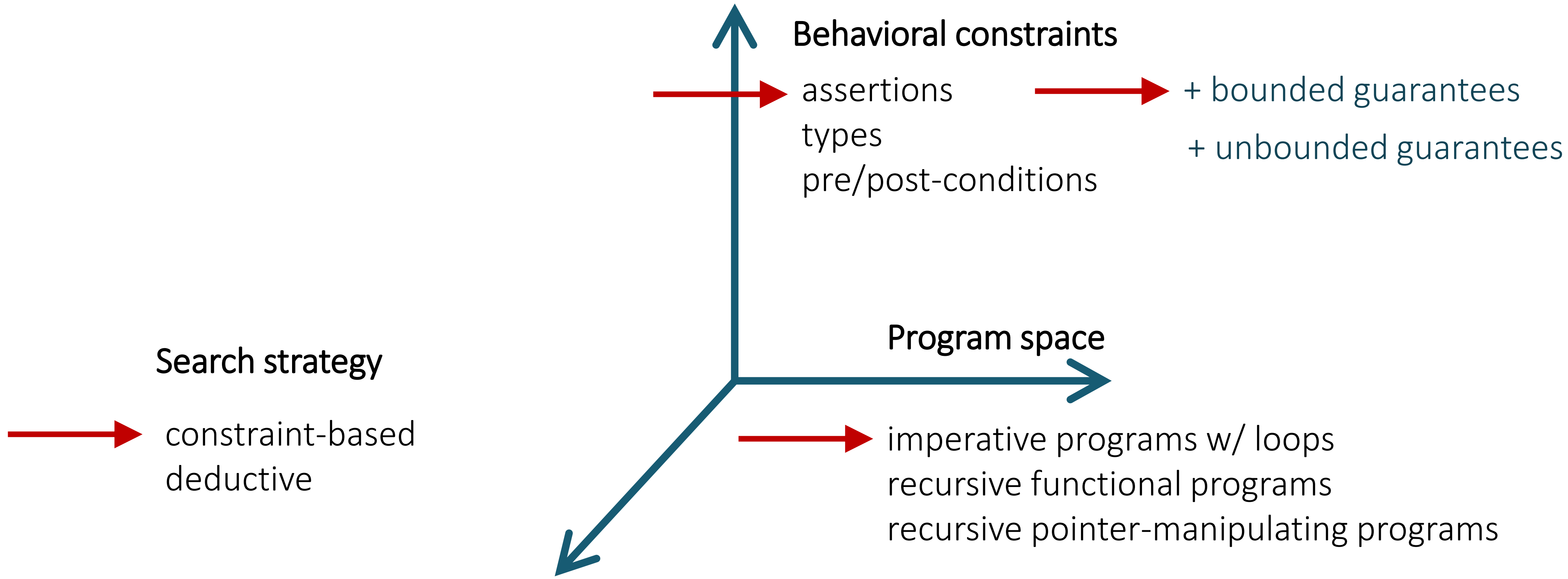
Program space

imperative programs w/ loops
recursive functional programs
recursive pointer-manipulating programs

Search strategy

constraint-based
deductive

Last week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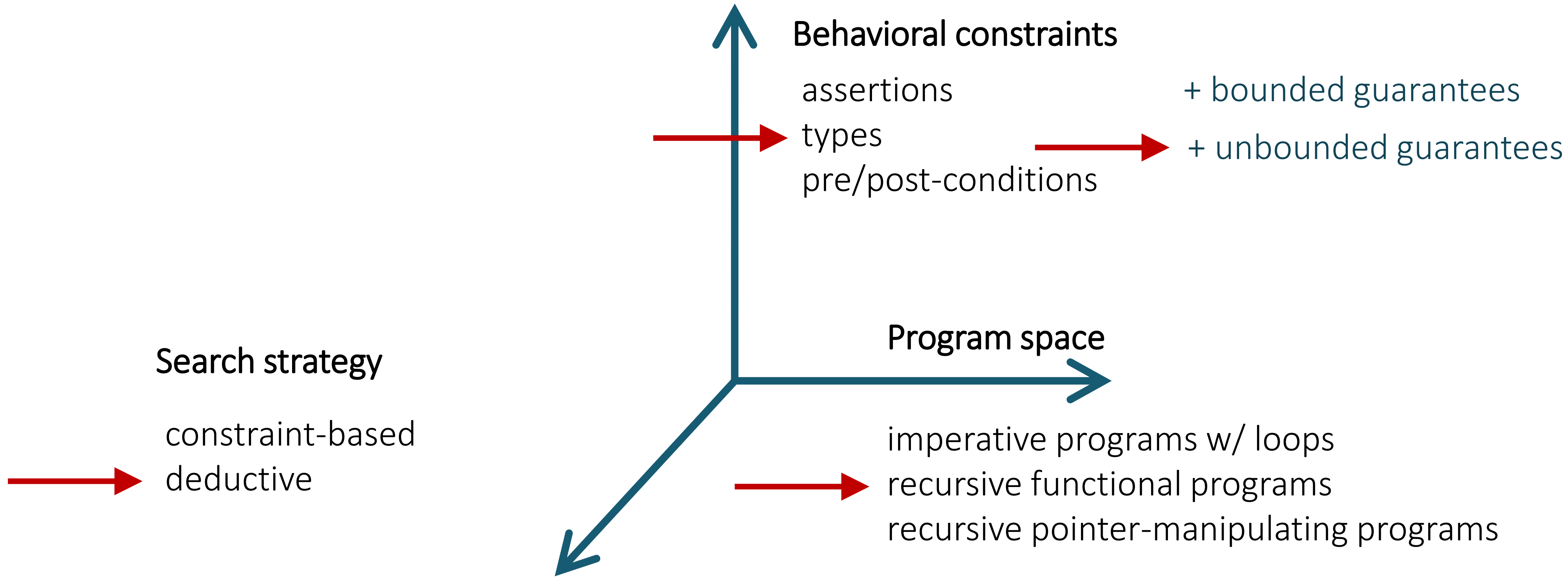
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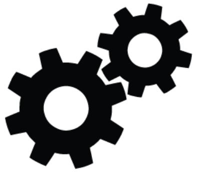
Type-driven program synthesis

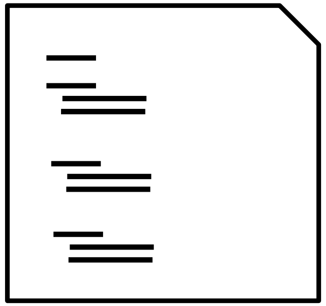
specification

programmer-friendly
informative

search



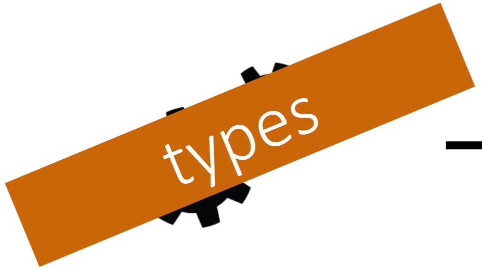
program



Type-driven program synthesis

specification

search



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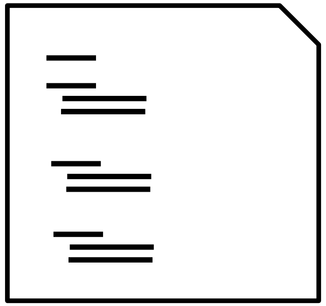


programmer-friendly
informative

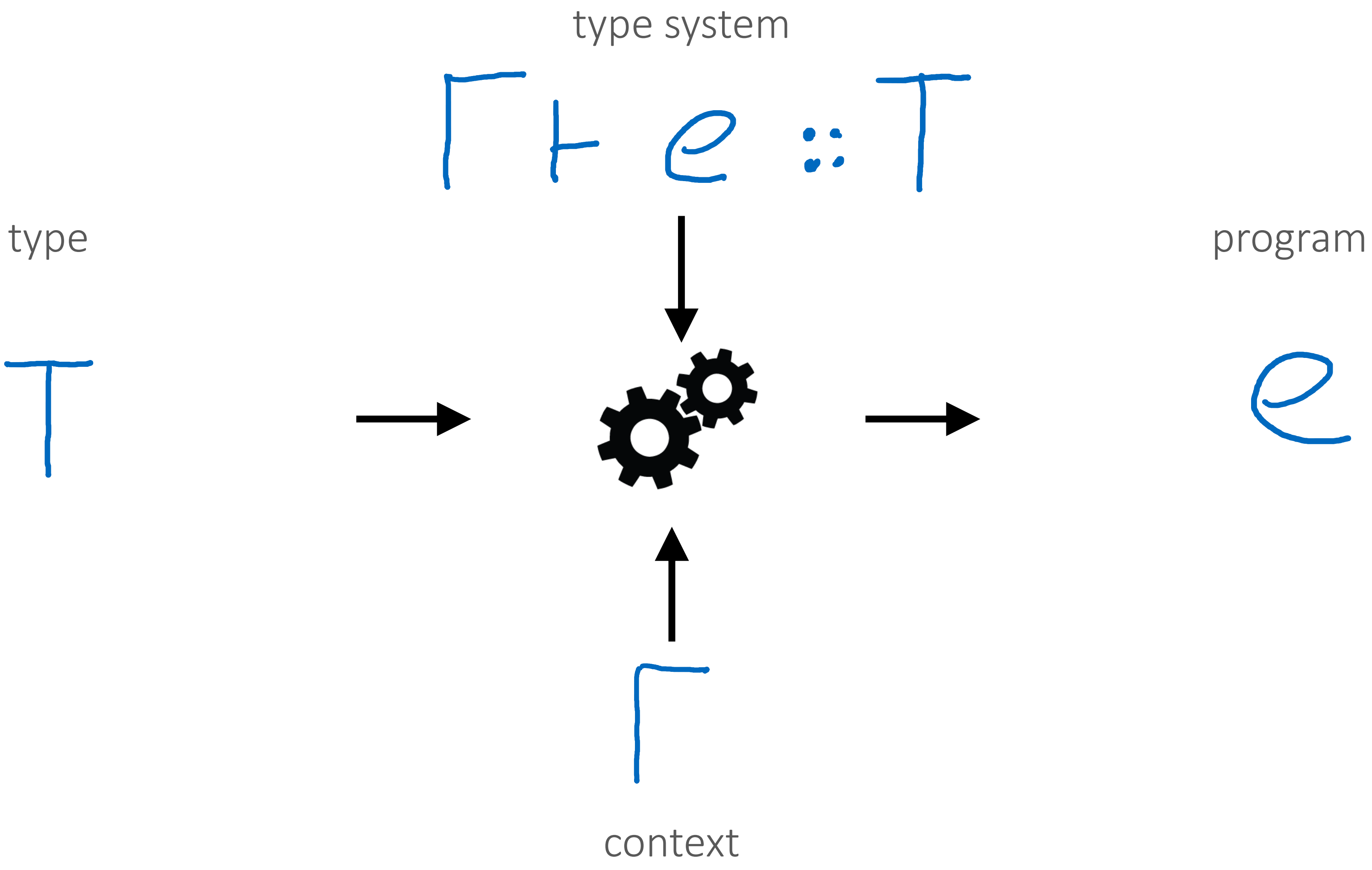
help prune
the search space

Which program do I have in mind?

`Char -> String -> [String]` split string at custom separator

`a -> Int -> [a]` list with n copies of input value

Type-driven program synthesis



This week

intro to type systems

enumerating well-typed terms

bidirectional type systems

synthesis with types and examples

polymorphic types

refinement types

synthesis with refinement types

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What is a type system?

Deductive system for proving facts about programs and types

Defined using *inference rules* over *judgments*

typing judgement

program / term

context → $\Gamma \vdash e :: T$ ← type

"under context Gamma, term e has type T"

A simple type system: syntax

$e ::= 0 \mid e+1 \mid x \mid e\ e \mid \lambda x.e$ -- expressions

example program: increment by two $\lambda x.(x+1)+1$

A simple type system: syntax

$e ::= 0 \mid e+1 \mid x \mid e\ e \mid \lambda x.e$ -- expressions

$T ::= \mathrm{Int} \mid T \to T$ -- types

$\Gamma ::= \cdot \mid x{:}T, \Gamma$ -- contexts

Inference rules = typing rules

$$\text{t-zero}\ \frac{}{\Gamma \vdash 0 :: \mathrm{Int}} \qquad \text{t-suc}\ \frac{\Gamma \vdash e :: \mathrm{Int}}{\Gamma \vdash e+1 :: \mathrm{Int}} \qquad \text{t-var}\ \frac{x : T \in \Gamma}{\Gamma \vdash x :: T}$$

$$\text{t-abs}\ \frac{\Gamma, x : T_1 \vdash e :: T_2}{\Gamma \vdash \lambda x. e :: T_1 \to T_2} \qquad \text{t-app}\ \frac{\Gamma \vdash e_1 :: T' \to T \qquad \Gamma \vdash e_2 :: T'}{\Gamma \vdash e_1\, e_2 :: T}$$

Typing derivations

A derivation of $\Gamma \vdash e :: T$ is a tree where

1. the root is $\Gamma \vdash e :: T$
2. children are related to parents via inference rules
3. all leaves are axioms

Typing derivations

let's build a derivation of

$$\cdot \vdash \lambda x.\, x + 1 :: \mathrm{Int} \rightarrow \mathrm{Int}$$

we say that $\lambda x.\, x + 1$ is well-typed in the empty context

and has type $\mathrm{Int} \rightarrow \mathrm{Int}$

Typing derivations

$$\frac{}{\Gamma \vdash 0 :: \text{Int}}\ \text{t-zero} \qquad \frac{\Gamma \vdash e :: \text{Int}}{\Gamma \vdash e+1 :: \text{Int}}\ \text{t-suc}$$

$$\frac{x : T \in \Gamma}{\Gamma \vdash x :: T}\ \text{t-var} \qquad \frac{\Gamma, x : T_1 \vdash e :: T_2}{\Gamma \vdash \lambda x. e :: T_1 \to T_2}\ \text{t-abs}$$

$$\frac{\Gamma \vdash e_1 :: T' \to T \qquad \Gamma \vdash e_2 :: T'}{\Gamma \vdash e_1\, e_2 :: T}\ \text{t-app}$$

$$\cdot \vdash \lambda x. x + 1 :: \text{Int} \to \text{Int}$$

Typing derivations

is $(\lambda x. x) + 1$ well-typed (in the empty context)?

no! no way to build a derivation of $\cdot \vdash (\lambda x. x) + 1 :: \_$

we say that $(\lambda x. x) + 1$ is ill-typed

Let's add lists!

$e ::= \ldots \mid [\,] \mid e : e \mid \text{match } e \text{ with } [\,] \to e \mid x : x \to e$

$T ::= \text{Int} \mid \text{List} \mid T \to T$

Example program: head with default

$$\lambda x.\,\mathrm{match}\ x\ \mathrm{with}\ nil \to 0 \mid y{:}\,ys \to y$$

Typing rules

$$\text{t-nil}\ \frac{}{\Gamma \vdash [\,] :: \mathit{List}}$$

$$\text{t-cons}\ \frac{\Gamma \vdash e_1 :: \mathit{Int} \qquad \Gamma \vdash e_2 :: \mathit{List}}{\Gamma \vdash e_1 : e_2 :: \mathit{List}}$$

what should the t-match tule be?

$$\text{t-match}\ \frac{\Gamma \vdash e_0 :: \boxed{}^{1} \qquad \Gamma \vdash e_1 :: \boxed{}^{2} \qquad \Gamma\ \boxed{}^{4} \vdash e_2 :: \boxed{}^{3}}{\Gamma \vdash \mathtt{match}\ e_0\ \mathtt{with}\ [\,] \to e_1 \mid x:xs \to e_2 :: T}$$

Typing rules

$$\text{t-nil}\ \frac{}{\Gamma \vdash [\,] :: \text{List}}$$

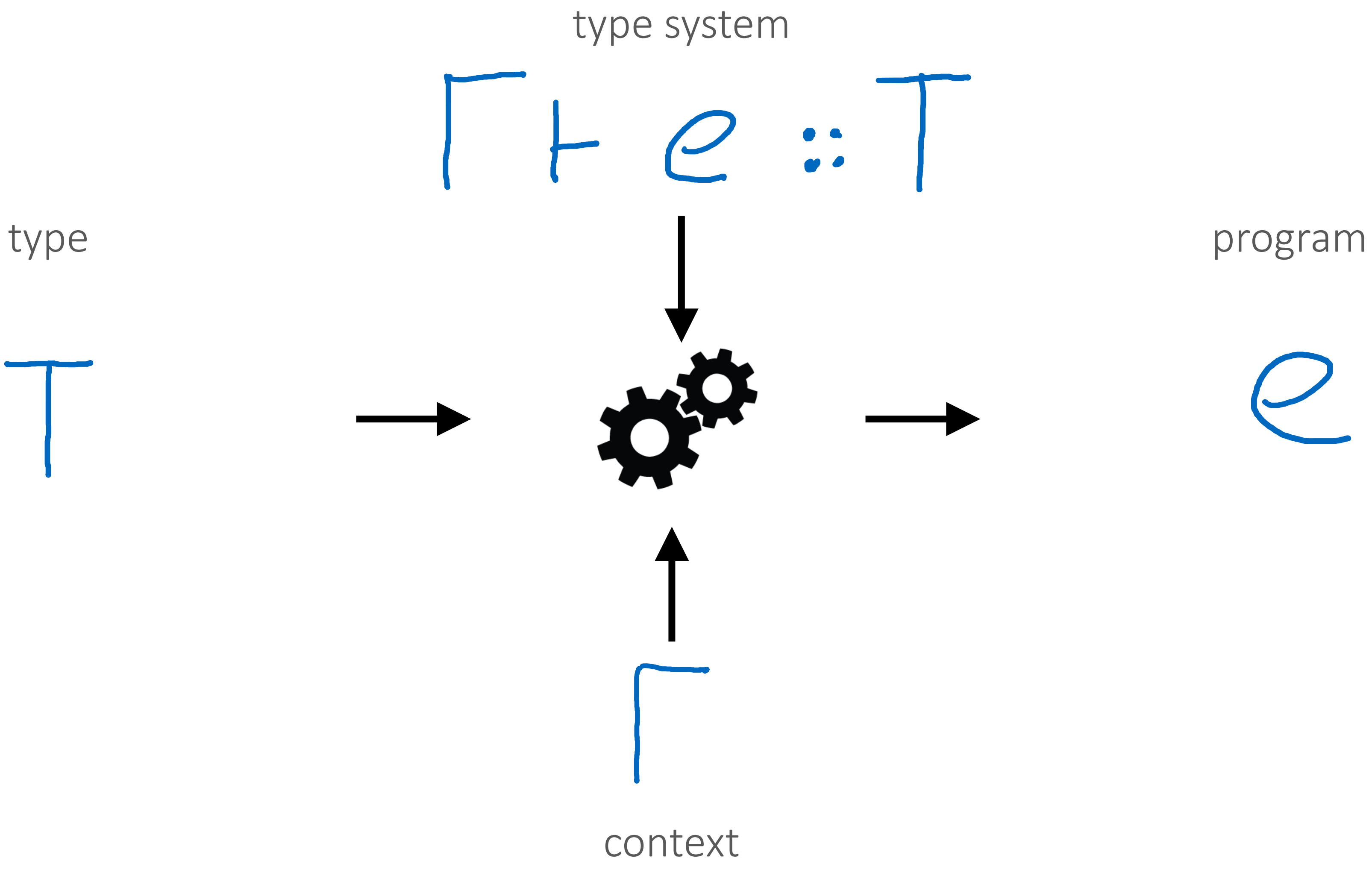
$$\text{t-cons}\ \frac{\Gamma \vdash e_1 :: \text{Int} \qquad \Gamma \vdash e_2 :: \text{List}}{\Gamma \vdash e_1 : e_2 :: \text{List}}$$

$$\text{t-match}\ \frac{\Gamma \vdash e_0 :: \text{List} \qquad \Gamma \vdash e_1 :: T \qquad \Gamma, x:\text{Int}, xs:\text{List} \vdash e_2 :: T}{\Gamma \vdash \text{match } e_0 \text{ with } [\,] \to e_1 \mid x:xs \to e_2 :: T}$$

Example: head with default

$$\cdot \vdash \lambda x.\, \text{match } x \text{ with } nil \to 0 \mid y : ys \to y \;\; :: \text{List} \to \text{Int}$$

Type system → synthesis



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Enumerating well-typed terms

how should I enumerate all terms of type $\mathrm{List} \to \mathrm{List}$?

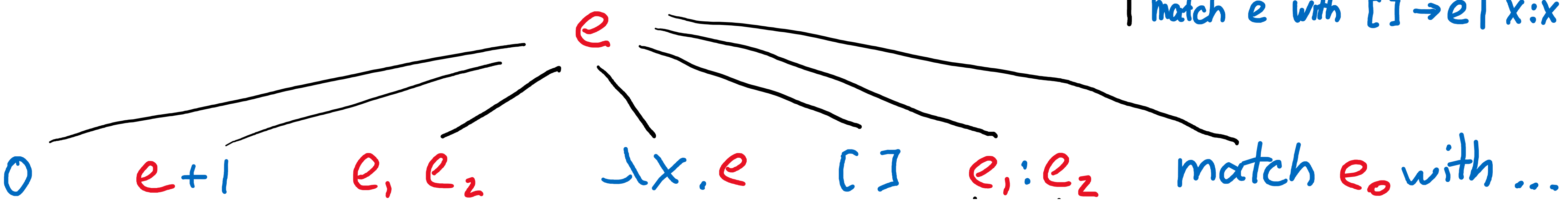
(up to depth 2, in the empty context)

naïve idea: syntax-guided enumeration

1. enumerate all terms *generated by the grammar*
2. type-check each term and throw away ill-typed ones

Syntax-guided enumeration

$$e ::= 0 \mid e+1 \mid x \mid e\ e \mid \lambda x.e \mid [\,] \mid e:e \mid \text{match } e \text{ with } [\,] \to e \mid x:x \to e$$



31 complete programs enumerated

only 2 have the type $\text{List} \to \text{List}$!

can we do better?

Enumerating well-typed terms

how should I enumerate all terms of type $\text{List} \to \text{List}$?

(up to depth 2, in the empty context)

better idea: type-guided enumeration

enumerate all derivations *generated by the type systems*

extract terms from derivations (well-typed by construction)

Synthesis as proof search

input: synthesis goal $\Gamma \vdash ? :: T$

output: derivation of $\Gamma \vdash e :: T$ for some e

search strategy: top-down enumeration of derivation trees

like syntax-guided top-down enumeration but

derivation trees instead of ASTs

typing rules instead of grammar

Type-guided enumeration

$$\text{t-var}\ \frac{x:T \in \Gamma}{\Gamma \vdash x :: T} \qquad \text{t-zero}\ \frac{}{\Gamma \vdash 0 :: Int} \qquad \text{t-succ}\ \frac{\Gamma \vdash e :: Int}{\Gamma \vdash e+1 :: Int}$$

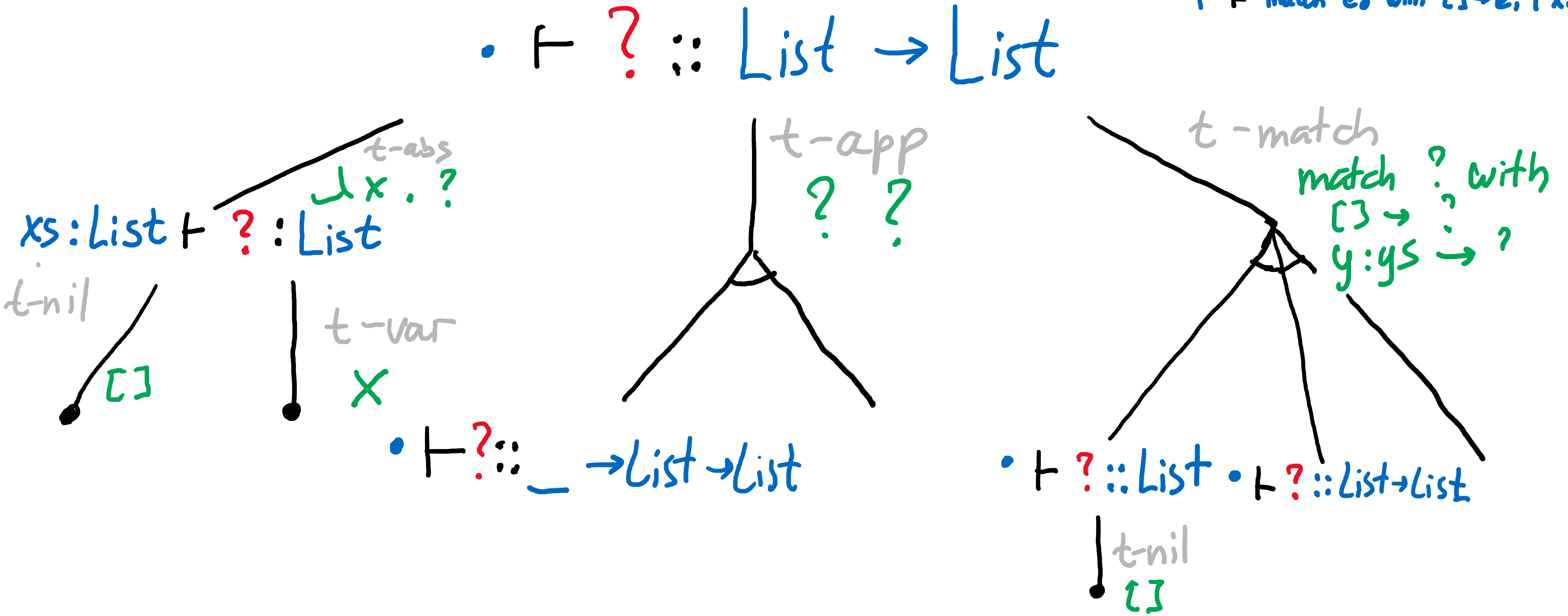
$$\text{t-abs}\ \frac{\Gamma, x:T_1 \vdash e :: T_2}{\Gamma \vdash \lambda x.e :: T_1 \to T_2} \qquad \text{t-app}\ \frac{\Gamma \vdash e_1 :: T' \to T \quad \Gamma \vdash e_2 :: T'}{\Gamma \vdash e_1\ e_2 :: T}$$

$$\text{t-nil}\ \frac{}{\Gamma \vdash [\,] :: List} \qquad \text{t-cons}\ \frac{\Gamma \vdash e_1 :: Int \quad \Gamma \vdash e_2 :: List}{\Gamma \vdash e_1 : e_2 :: List}$$

$$\text{t-match}\ \frac{\Gamma \vdash e_0 :: List \quad \Gamma \vdash e_1 :: T \quad \Gamma, x:Int, xs:List \vdash e_2 :: T}{\Gamma \vdash \mathtt{match}\ e_0\ \mathtt{with}\ [\,] \to e_1 \mid x:xs \to e_2 :: T}$$

only 2 programs fully constructed!

all other programs *rejected early*



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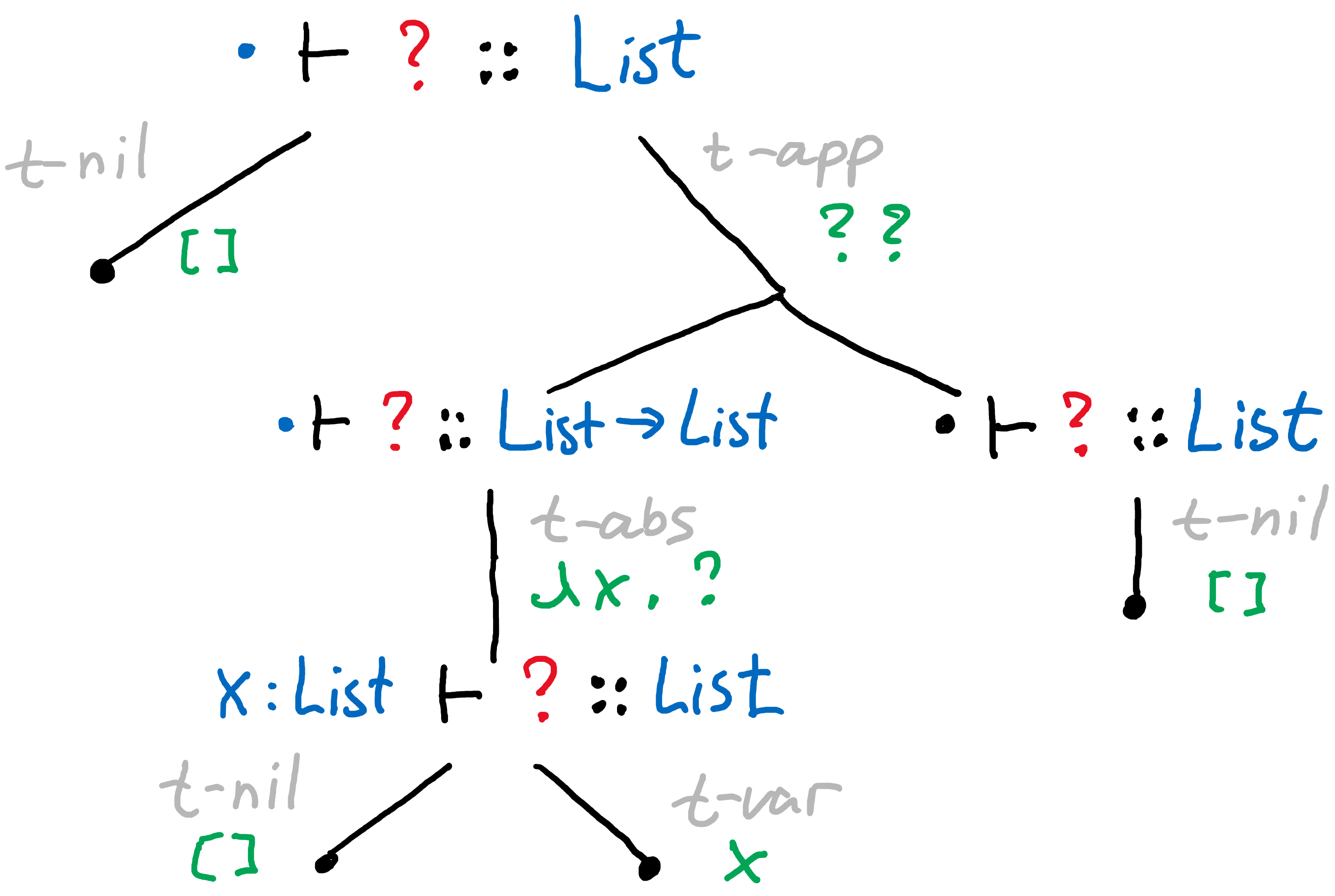
Bidirectional type system

Makes top-down propagation of types explicit

Helps with equivalence reduction

What's wrong with this search?



Enumerated 3 programs:

$$nil$$
$$(\lambda x. x)\ nil$$
$$(\lambda x. nil)\ nil$$

They are all equivalent!

Redundant programs

$$(\lambda x.\, e_1)\; e_2 \xrightarrow{\beta\text{-reduction}} e_1\,[x := e_2]$$

$$\begin{array}{l}\textsf{match } [\,] \textsf{ with} \\ \quad [\,] \to e_1 \\ \quad y{:}ys \to e_2\end{array} \xrightarrow{\text{match-nil}} e_1$$

$$\begin{array}{l}\textsf{match } e_1 : e_2 \textsf{ with} \\ \quad [\,] \to e_3 \\ \quad y{:}ys \to e_4\end{array} \xrightarrow{\text{match-cons}} e_4\,[y := e_1,\, ys := e_2]$$

Generating programs on the left is a waste of time!

Idea: only generate programs *in normal form*

Restrict type system to make redundant programs ill-typed

Normal-form programs

$$e ::= x \mid e\ i$$

elimination forms

$$i ::= 0 \mid i+1 \mid \lambda x.\, i \mid [\,] \mid i : i$$
$$\mid \text{match } e \text{ with } [\,] \to i \mid x : x \to i$$

introduction forms

$$B ::= \text{Int} \mid \text{List}$$

base types

$$T ::= B \mid T \to T$$

types

Bidirectional typing judgments

$\Gamma \vdash i \Leftarrow T$ “under context Gamma, i checks against type T”

$\Gamma \vdash e \Rightarrow T$ “under context Gamma, e generates type T”

[Pierce, Turner. Local Type Inference. 2000]

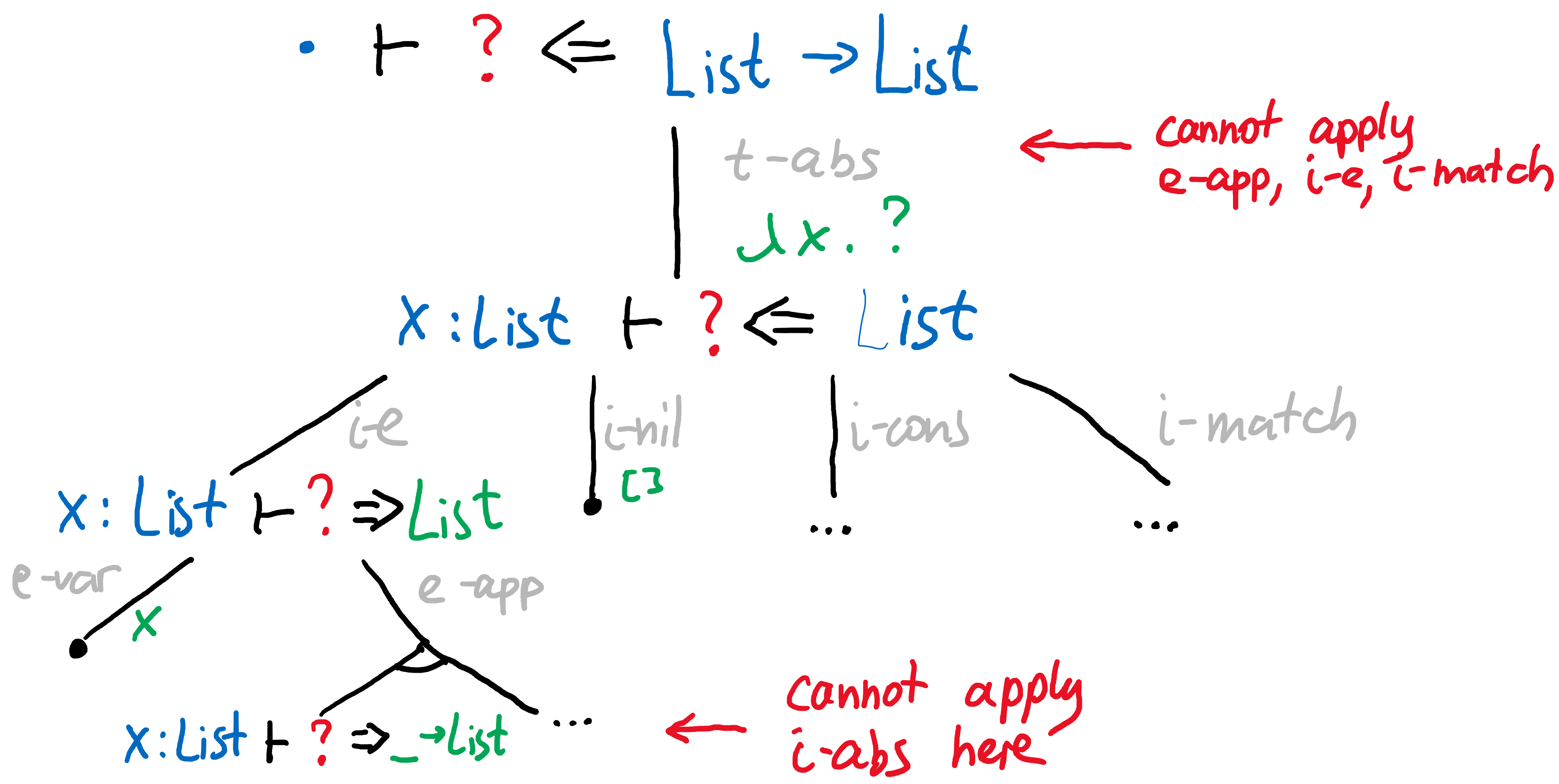
Bidirectional typing rules

$$\text{e-var}\ \frac{x : T \in \Gamma}{\Gamma \vdash x \Rightarrow T} \qquad \text{e-app}\ \frac{\Gamma \vdash e \Rightarrow T' \to T \quad \Gamma \vdash i \Leftarrow T'}{\Gamma \vdash e\ i \Rightarrow T}$$

$$\text{i-e}\ \frac{\Gamma \vdash e \Rightarrow B}{\Gamma \vdash e \Leftarrow B} \qquad \text{i-nil}\ \frac{}{\Gamma \vdash [\,] \Leftarrow List} \qquad \text{i-cons}\ \frac{\Gamma \vdash i_1 \Leftarrow Int \quad \Gamma \vdash i_2 \Leftarrow List}{\Gamma \vdash i_1 : i_2 \Leftarrow List}$$

$$\text{i-abs}\ \frac{\Gamma, x : T_1 \vdash i \Leftarrow T_2}{\Gamma \vdash \lambda x. i \Leftarrow T_1 \to T_2} \qquad \text{i-match}\ \frac{\Gamma \vdash e \Rightarrow List \quad \Gamma \vdash i_1 \Leftarrow B \quad \Gamma, y{:}Int, ys{:}List \vdash i_2 \Leftarrow B}{\Gamma \vdash \text{match } e \text{ with } [\,] \Rightarrow i_1 \mid y{:}ys \Rightarrow i_2 \Leftarrow B}$$

Type-guided enumeration



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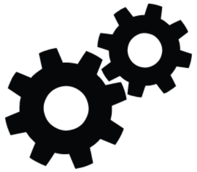
Simple types are not enough

specification

code

"duplicate every element in a list"

`stutter :: List → List` ⟶



⟶ $\lambda xs.\, xs$

Simple types are not enough

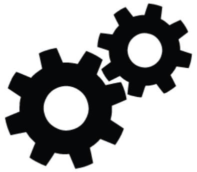
specification

code

“insert element
into so l list”

ambiguous!

```
insert
  In   List → List
```



$$\lambda x. \lambda xs. xs$$

Type-driven synthesis in 3 easy steps

1. Annotate types with extra specs
   examples, logical predicates, resources, …
2. Design a type system for annotated types
   propagate as much info as possible from conclusion to premises
3. Perform type-directed enumeration as before

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Type + examples

specification

code

"duplicate every element in a list"

List → List

$\triangleright \langle [\,] \to [\,] \mid [0] \to [0,0] \mid [1,0] \to [1,1,0,0] \rangle$

Myth

$$
\begin{aligned}
&\text{fix } f\,(xs).\,\text{match } xs \text{ with}\\
&\quad nil \to nil\\
&\mid\ y{:}\,ys \to y{:}\,y{:}\,f\,(ys)
\end{aligned}
$$

[Osera, Zdancewic , Type-and-Example-Directed Program Synthesis. 2015]

Types + examples: syntax

partial function

$V ::= 0 \mid V+1 \mid [\,] \mid V:V \mid \overline{V \to V}$

values

$X ::= \begin{matrix} V_1 \\ \vdots \\ V_n \end{matrix}$

vectors of examples

$R ::= T \triangleright X$

type refined with examples

$\Gamma ::= \cdot \mid x:R, \Gamma$

context

Example: singleton

$\bullet \vdash \mathit{Int} \to \mathit{List} \triangleright \langle 0 \to [0] \mid 1 \to [1] \rangle \leadsto \lambda x.\ x:[]$

abs

$x:\mathit{Int} \vdash \mathit{List} \leadsto x:[]$

cons

$x:\mathit{Int} \triangleright {0 \atop 1} \vdash \mathit{Int} \leadsto x$ $\qquad$ $x:\mathit{Int} \triangleright {0 \atop 1} \vdash \mathit{List} \leadsto []$

no search! simply propagate the spec top-down

Type-driven synthesis in 3 easy steps

1. Annotate types with examples
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3. Perform type-directed enumeration as before

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Polymorphic types

$$\forall \alpha .\ \alpha \to \mathrm{List}\ \alpha$$

Polymorphic types for synthesis

- $\vdash ? :: \text{Int} \to \text{List Int}$
- $\vdash ? :: \forall \alpha.\ \alpha \to \text{List}\ \alpha$

$\lambda x.\, nil$

$\lambda x.\,[0], \lambda x.\,[1], \ldots$

$\lambda x.\,[x]$

$\lambda x.\,[\text{double } 0], \lambda x.\,[\text{dec } 0]$

$\lambda x.\,[0{,}0], \lambda x.\,[0{,}1], \ldots$

$\lambda x.\,[x, x]$

which of these programs match the polymorphic type?

Polymorphic types for synthesis

- $\vdash\ ?\ ::\ \mathit{Int} \rightarrow \mathit{List}\ \mathit{Int}$

$\lambda x.\mathit{nil}$

$\lambda x.[0], \lambda x.[1], \ldots$

$\lambda x.[x]$

$\lambda x.[\mathrm{double}\ 0], \lambda x.[\mathrm{dec}\ 0]$

$\lambda x.[0{,}0], \lambda x.[0{,}1], \ldots$

$\lambda x.[x, x]$

- $\vdash\ ?\ ::\ \forall\alpha.\ \alpha \rightarrow \mathit{List}\ \alpha$

1. $\lambda x.\mathit{nil}$

eliminate ambiguity!

2. $\lambda x.[x]$

prune the search!

3. $\lambda x.[x, x]$

Polymorphic types

$$B ::= \mathrm{Int} \mid \mathrm{List}\ B \mid \alpha$$ base types

$$T ::= B \mid T \to T$$ types

$$S ::= T \mid \forall \alpha . S$$ type schemas (polytypes)

$$\Gamma ::= \cdot \mid x : S, \Gamma \mid \alpha, \Gamma$$ contexts

Judgments

$\Gamma \vdash i \Leftarrow S$

type checking:

"under context Gamma, i checks against a schema S"

$\Gamma \vdash e \Rightarrow T$

type inference:

"under context Gamma, e generates type T"

Typing rules

$$\frac{}{\Gamma \vdash [\,] \Leftarrow \text{List } B}\ \textit{i-nil}$$

$$\frac{\Gamma \vdash i_1 \Leftarrow B \quad \Gamma \vdash i_2 \Leftarrow \text{List } B}{\Gamma \vdash i_1 :: i_2 \Leftarrow \text{List } B}\ \textit{i-cons}$$

$$\frac{\Gamma, \alpha \vdash i \Leftarrow S}{\Gamma \vdash i \Leftarrow \forall \alpha . S}\ \textit{i-gen}$$

$$\frac{x : \forall \overline{\alpha} . T \in \Gamma}{\Gamma \vdash x \Rightarrow \overline{[\alpha \mapsto T']}\, T}\ \textit{e-var}$$

how do we guess T'?

Hindley-Milner type inference!

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Refinement types

```
Nat
```

base types

```
max :: x: Int → y: Int → { v: Int | x ≤ v ∧ y ≤ v }
```

dependent function types

```
xs :: { v: List Nat }
```

polymorphic datatypes

[Rondon, Kawaguchi, Jhala. PLDI'2008]

Refinement types: measures

```
data List α where
    Nil   ::  { List α | len v = 0 }
    Cons   ::   x: α   → xs: List α
                            → { List α | len v = len xs + 1 }
```

syntactic sugar:

```
measure len :: List α → Int
    len Nil = 0
    len (Cons _ xs) = len xs + 1
```

example: duplicate every element in a list

```
stutter :: ??
```

Refinement types: sorted lists

```
data SList α where
    Nil   ::  SList α
    Cons   ::  x: α  → xs: SList {α | x ≤ ν }
                     → SList α
```

example: insert an element into a sorted list

```
insert :: ??
```

Refinement types

$$B ::= \text{Int} \mid \text{List } B \mid \alpha$$

base types

$$T ::= \{\nu{:}B \mid \varphi\} \mid x{:}T \to T$$

types

$$S ::= T \mid \forall \alpha . S$$

type schemas (polytypes)

$$\Gamma ::= \cdot \mid x{:}S, \Gamma \mid \alpha, \Gamma$$

contexts

Example: increment

$$\text{Nat} = \{\nu : \text{Int} \mid \nu \geq 0\}$$

$$\Gamma = [\text{inc}: y: \text{Int} \to \{\nu: \text{Int} \mid \nu = y + 1\}]$$

we need subtyping!

$$\Gamma \vdash \lambda x.\, \text{inc}\ x \Leftarrow \text{Nat} \to \text{Nat}$$

Subtyping

intuitively: T' is a subtype of T if all values of type T' also belong to T

written $T' <: T$

e.g. $\mathrm{Nat} <: \mathrm{Int}$ or $\{\nu{:}\,\mathrm{Int} \mid \nu = 5\} <: \mathrm{Nat}$

sub-base $$\frac{[\![\Gamma]\!] \wedge \phi' \Rightarrow \phi}{\Gamma \vdash \{\nu{:}\ B \mid \phi'\} <: \{\nu{:}\ B \mid \phi\}}$$

sub-fun $$\frac{\Gamma \vdash T_1 <: T_1' \qquad \Gamma; x{:}\,T_1 \vdash T_2' <: T_2}{\Gamma \vdash x{:}\,T'_1 \to T_2' <: x{:}\,T_1 \to T_2}$$

$\mathrm{Pos} <: \mathrm{Nat}$ ✓

$\mathrm{Int} \to \mathrm{Int} <: \mathrm{Int} \to \mathrm{Nat}$ ✗

$\mathrm{Int} \to \mathrm{Int} <: \mathrm{Nat} \to \mathrm{Int}$ ✓

$x{:}\mathrm{Int} \to \{\mathrm{Int} \mid \nu = x + 1\} <: \mathrm{Nat} \to \mathrm{Nat}$ ✓

Typing rules

$$\textit{i-e}\ \frac{\Gamma \vdash e \Rightarrow T \qquad \Gamma \vdash T <: \{B \mid \varphi\}}{\Gamma \vdash e \Leftarrow \{B \mid \varphi\}}$$

$$\textit{e-app}\ \frac{\Gamma \vdash e \Rightarrow y{:}T_1 \to T_2 \qquad \Gamma \vdash i \Leftarrow T_1}{\Gamma \vdash e\ i \Rightarrow [y \mapsto i]\,T_2}$$

Example: increment

$$\Gamma = [\mathrm{inc}\colon y\colon \mathrm{Int} \to \{\nu\colon \mathrm{Int} \mid \nu = y + 1\}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, x{:}\mathrm{Nat} \vdash \mathrm{inc} \Rightarrow y{:}\mathrm{Int} \to \{\mathrm{Int} \mid \nu = y+1\}}\,\text{e-var} \quad \dfrac{\dfrac{}{\Gamma, x{:}\mathrm{Nat} \vdash x \Rightarrow \mathrm{Nat}}\,\text{e-var} \quad \Gamma, x{:}\mathrm{Nat} \vdash \mathrm{Nat} <: \mathrm{Int}}{\Gamma, x{:}\mathrm{Nat} \vdash x \Leftarrow \mathrm{Int}}\,\text{i-e}}{\Gamma, x{:}\mathrm{Nat} \vdash \mathrm{inc}\ x \Rightarrow \{\mathrm{Int} \mid \nu = x+1\}}\,\text{e-app} \quad \Gamma, x{:}\mathrm{Nat} \vdash \{\mathrm{Int} \mid \nu = x+1\} <: \mathrm{Nat}}{\Gamma, x{:}\mathrm{Nat} \vdash \mathrm{inc}\ x \Leftarrow \mathrm{Nat}}\,\text{i-e}}{\Gamma \vdash \lambda x.\ \mathrm{inc}\ x \Leftarrow \mathrm{Nat} \to \mathrm{Nat}}\,\text{i-abs}$$

subtyping constraints

$$\mathrm{Nat} <: \mathrm{Int}$$

$$x\colon \mathrm{Nat} \vdash \{\nu\colon \mathrm{Int} \mid \nu = x + 1\} <: \mathrm{Nat}$$

implications

$$\nu \geq 0 \Rightarrow \mathit{true}$$

$$x \geq 0 \wedge \nu = x + 1 \Rightarrow \nu \geq 0$$

SMT solver: VALID!

Refinement type checking

idea: separate type checking into
subtyping constraint generation and subtyping constraint solving

1. Generate a constraint for every subtyping premise in derivation
2. Reduce subtyping constraints to implications
3. Use SMT solver to check implications

[Rondon, Kawaguchi, Jhala. PLDI'2008]

This week

intro to type systems

enumerating well-typed terms

bidirectional type systems

synthesis with types and examples

polymorphic types

refinement types

synthesis with refinement types

Synthesis from refinement types

specification

“duplicate every element in a list”

```
stutter ::
  xs:List a → 
  {ν:List a | len ν =
    2 * len xs}
```

code

```
match xs with
 Nil → Nil
 Cons h t →
  Cons h (Cons h (stutter t))
```

[Polikarpova, Kuraj, Solar-Lezama, Program Synthesis from Polymorphic Refinement Types. 2016]

Synthesis from refinement types

specification

“insert element
into sorted list”

```
insert :: x:a → 
  xs:SList a → 
  {ν:SList a | elems ν = 
    elems xs ∪ {x}}
```

code

```
match xs with
 Nil → Cons x Nil
 Cons h t → 
   if x ≤ h
     then Cons x xs
     else Cons h (insert x t)
```

Type-driven synthesis in 3 easy steps

1. Annotate types with logical predicates
2. Design a type system for annotated types
3. Perform type-directed enumeration as before

Type-directed enumeration for insert

```
x:a  →  xs:SList a  →
  {ν:SList a | elems ν = elems xs ∪ {x}}
```

↕

```
insert = ??
```

Type-directed enumeration

```
{ν:SList a | elems ν = elems xs ∪ {x}}
```

↕

```
insert x xs = ??
```

context:

```
x: a
xs: SList a
```

Type-directed enumeration

```
{ν:SList a | elems ν = elems xs ∪ {x}}
```

↕

```
insert x xs =
  match xs with
    Nil → ??
    Cons h t → ??
```

context:

```
x: a
xs: SList a
```

Type-directed enumeration

$\{\nu\text{:SList a} \mid \text{elems } \nu = \text{elems xs} \cup \{x\}\}$

$\updownarrow$

```
insert x xs =
  match xs with
    Nil → ??
    Cons h t → ??
```

context:
x: a
xs: SList a
elems xs = {}

Type-directed enumeration

```
{ν:SList a | elems ν = elems xs ∪ {x}}
```

```
insert x xs =
  match xs with
    Nil → Nil
    Cons h t → ??
```

context:
```
x: a
xs: SList a
elems xs = {}
```

Constraints:
```
∀x: {} = {} ∪ {x}
```

SMT solver: INVALID!

The hard part: application

```
x:a  →  xs:SList a  →
  {ν:SList a | elems ν = elems xs ∪ {x}}
```

↕

```
insert x xs =
  match xs with
    Nil → Cons x Nil
    Cons h t →
      Cons h (insert x ??)
```

should this program be rejected?

yes!
cannot guarantee output is sorted!

Round-trip type-checking (RTTC)

$\Gamma \vdash i \Leftarrow S$

type checking:

"under context Gamma, i checks against schema S"

$\Gamma \vdash e \Leftarrow T \Rightarrow T'$

type strengthening:

"under context Gamma, e checks against type T *and* generates a stronger type T'"

RTTC rules

$$\textit{i-e}\ \frac{\Gamma \vdash e \Leftarrow \{B \mid \varphi\} \Rightarrow T}{\Gamma \vdash e \Leftarrow \{B \mid \varphi\}}$$

$$\textit{e-var}\ \frac{x : T' \in \Gamma \qquad \Gamma \vdash T' <: T}{\Gamma \vdash x \Leftarrow T \Rightarrow T'}$$

$$\textit{e-app}\ \frac{\Gamma \vdash e \Leftarrow \bot \to T \Rightarrow y : T_1 \to T_2 \qquad \Gamma \vdash i \Leftarrow T_1}{\Gamma \vdash e\ i \Leftarrow T \Rightarrow [y \mapsto i]\, T_2}$$

The hard part: application

elems will depend on the missing part…

but sortedness we can already check!

```
{ν:SList a | elems ν = elems xs ∪ {x}}
```

```
insert x xs =
  match xs with
    Nil → Cons x Nil
    Cons h t →
      Cons h (insert x ??)
```

The hard part: application

$\{\nu{:}a \mid h \le \nu\}$

↕

```
insert x xs =
  match xs with
    Nil → Cons x Nil
    Cons h t →
      Cons h (insert x ??)
```

context:
x: a
xs: SList a
h: a
t: SList {a|h≤ν}

Constraints:
∀x, h: h ≤ x

SMT solver: INVALID!

```
insert :: x:τ →
  xs:SList τ →
  SList τ
```